in (36) and (37) leads to more accurate results in and near the operating band of the transformer than would the exact formula for $\rho|_{\phi=0}$.

Appendix II

The following simplified method of calculating the a_m values was developed for antenna-array applications by Ross E. Graves in an as yet unpublished report. It is adapted here with his permission for the stepped-transformer case.

TABLE IV
Computation of Relative a_m Values for $p=1.40$

$n=1$	2					
$n=2$		3.864				
$n=3$	27.861		14.930			
$n=4$		161.48		57.690		
etc.						

To employ Graves' method, it is necessary to construct a numerical table by a simple recursion procedure. To illustrate the method, a typical table is given above in Table IV for the case of $p=1.40$, $\phi_1=75.0$ degrees. In the upper left-hand corner always insert the number two for any value of p. In the second column, second row, always insert

$$x_0 = \frac{1}{\cos \phi_1}.$$

For this example, $x_0=1/\cos 75$ degrees$=3.864$. Then fill in the table by means of the following rules until the desired value of n is reached.

1. To find an additional entry in the first column, multiply the element on the right just above by $2x_0$ and then subtract the element in the second row above the entry to be found.
2. To find an additional entry in any other column, add the two elements on the left and right just above and multiply by x_0, and then subtract the element in the second row above the entry to be found.
3. Where an element is absent, assume it to be zero.

The illustrative table has been filled up to $n=4$. The elements in the table are in the ratio of the a_m constants, the first column corresponding to the center of the transformer. For example, for $n=3$,

$$a_1:a_2:a_3 = 14.930:27.861:14.930 = 1:1.8661:1$$

and for $n=4$,

$$a_1:a_2:a_3:a_4 = 57.690:161.48:161.48:57.690$$
$$= 1:2.799:2.799:1.$$

The table could be carried, if desired, to any value of n, no matter how large.

The Use of Scattering Matrices in Microwave Circuits

E. W. MATTHEWS, JR.†

Summary—Difficulties arising from the use of the impedance concept in microwave circuitry have led to the introduction of the scattering representation for work at these frequencies. This paper presents a development of the scattering approach in terms of fundamental transmission-line phenomena. The physical meaning of the quantities involved is brought out wherever possible and the relationships among the various elements of the scattering matrix are given. Several examples of the application of scattering techniques to analysis of the properties of microwave junctions are presented, and methods for measuring scattering parameters of such junctions are outlined.

Introduction

In conventional circuit theory, the fundamental quantities of interest are voltages and currents, and the parameters used to express relationships between them are called impedances or admittances. A single two-terminal circuit element may be characterized by a complex impedance, representing the ratio between the voltage and the current at its two terminals. The real part of this impedance (resistance) is related to the power dissipated in the circuit element, while the imaginary part (reactance) is a measure of the average energy stored in the element.

More complicated multi-terminal networks may be represented at a given frequency by an "equivalent circuit" consisting of a number of simple two-terminal elements in certain combinations or configurations, such as equivalent tee, pi, or ladder networks. The properties of such networks may alternatively be described in terms of generalized impedance (or admittance) relationships between terminals (or "ports," as currently named). This description is better understood generally in terms of the "self" and "mutual" impedances commonly used in coupled-circuit analysis as well as the "transfer" impedances appearing in vacuum-tube circuitry.

At microwave frequencies, certain difficulties are encountered in the application of conventional low-frequency circuit analysis techniques. As circuit dimen-

† Sperry Gyroscope Company, Great Neck, N. Y.

sions become comparable to the wavelength, currents at opposite ends of a single impedance element begin to differ in magnitude and phase, and it becomes difficult to specify the scalar potential difference between two terminals uniquely, so that the very definition of an impedance becomes ambiguous. This difficulty is circumvented to some extent by the use of transmission-line theory, which in essence takes account of longitudinal variations in current and voltage, while restricting transverse dimensions to small fractions of a wavelength, as usually assumed for coaxial or two-wire lines. The treatment of hollow waveguides progresses one additional step, in seeking to account for the transverse distribution of currents and electric and magnetic fields from fundamental electromagnetic theory. Furthermore, the existence of higher "modes" in the region of discontinuities on transmission lines, such as are inevitably associated with terminating impedances and other circuit elements, requires the use of characteristic reference planes located some distance from the actual discontinuities (usually one or more half-wavelengths). The specification of the properties of a microwave network in terms of impedances or admittances in the face of such difficulties is at best laborious, and certainly tends to obscure the more important properties amid the algebraic relations which ensue. One is thus led to look for simpler and more refined analysis techniques for handling microwave circuits; such techniques are provided naturally by the use of the scattering representation.

The Scattering Representation

Scattering coefficients were apparently first mentioned by Campbell and Foster[1] in 1922, and have recently been more completely exploited for microwave and transmission-line problems,[2,3] as well as for general network theory.[4] Their use grew naturally from a physical interpretation of one solution to the standard transmission-line differential equations for the voltage and current as a function of distance along such a line. These equations are:

$$\frac{d^2V}{dX^2} = \gamma^2 V \qquad \frac{d^2I}{dX^2} = \gamma^2 I \tag{1}$$

and, with the usual harmonic time dependence, the solutions are:

$$V(x, t) = Ae^{-\gamma x + j\omega t} + Be^{\gamma x + j\omega t}$$
$$I(x, t) = \frac{1}{Z_c}\left(Ae^{-\gamma x + j\omega t} - Be^{\gamma x + j\omega t}\right), \tag{2}$$

where γ is the complex propagation constant, made up of a real part α, known as the attenuation constant, and an imaginary part β, called the phase factor and equal numerically to $2\pi/\lambda \cdot Z_c$ is known as the characteristic impedance of the transmission line, and is related to its physical dimensions. Separating the exponential factor in (2) into real and imaginary parts yields:

$$V(x, t) = Ae^{-\alpha x}e^{j(\omega t - \beta x)} + Be^{\alpha x}e^{j(\omega t + \beta x)}$$
$$Z_cI(x, t) = Ae^{-\alpha x}e^{j(\omega t - \beta x)} - Be^{\alpha x}e^{j(\omega t + \beta x)}. \tag{3}$$

It is apparent from an examination of the phase fronts, represented by $(\omega t \pm \beta x) =$ constant, that this solution represents a pair of waves traveling in opposite directions on the transmission line with a velocity $v = \omega/\beta$ and an exponential attenuation, or decrease in amplitude, in the direction of propagation.

These two traveling waves may alternatively be chosen as the independent variables for the transmission line problem, and defined as follows:

$$A(x, t) = Ae^{-\gamma x + j\omega t} = V(x, t) + Z_cI(x, t)$$
$$B(x, t) = Be^{\gamma x + j\omega t} = V(x, t) - Z_cI(x, t). \tag{4}$$

In general, A and B will both be complex quantities because of the arbitrary phase relations which may exist between V and I. $A(x, t)$ may be identified as the component wave traveling in the $+x$ direction, and $B(x, t)$ as the wave traveling in the $-x$ direction; both have the dimensions of a voltage.

For reasons which will appear later, it is more convenient to use a normalized form for component waves; this normalization is usually on a power basis. Thus if we consider a line with matched termination at $x = s$, so

$$\frac{V(s, t)}{I(s, t)} = Z_c,$$

it may be seen from (4) that $B(x, t) = 0$. The power dissipated in the matched termination is given by:

$$P = \frac{|V(s, t)|^2}{2Z_c} = \frac{|A(x, t)|^2}{8Z_c} \tag{5}$$

(the factor $\frac{1}{2}$ is necessary since we are essentially dealing with peak values). It is evident that a wave of given amplitude (or voltage) thus represents a rate of power flow which depends upon the characteristic impedance Z_c of the line on which it exists. In order to avoid this situation, we need only redefine the component waves in terms of the power which they represent, i.e., set $P = \frac{1}{2}|a(x, t)|^2$ in (5). Thus we define the normalized component waves as follows:

$$a(x, t) = \frac{1}{2}\left[\frac{V(x, t)}{\sqrt{Z_c}} + \sqrt{Z_c}I(x, t)\right]$$
$$b(x, t) = \frac{1}{2}\left[\frac{V(x, t)}{\sqrt{Z_c}} - \sqrt{Z_c}I(x, t)\right]. \tag{6}$$

Consequently, the power being propagated in the $+x$ direction is given simply by $\frac{1}{2}|a|^2 = (\frac{1}{2})aa^*$ (a^* is the complex conjugate of a), and the power being propa-

[1] G. A. Campbell and R. M. Foster, "Maximum output network for telephone substation and repeater circuits," *Trans. AIEE*, vol. 39, pp. 231–280; 1920.

[2] C. G. Montgomery, R. H. Dicke, and E. M. Purcell, "Principles of Microwave Circuits," Radiation Lab. Ser., vol. 8, McGraw-Hill Book Co., Inc., New York, N. Y., 1947.

[3] N. Marcuvitz, "The Waveguide Handbook," Radiation Lab. Ser., vol. 10, McGraw-Hill Book Co., Inc., New York, N. Y.; 1951.

[4] H. J. Carlin, "An Introduction to the Use of the Scattering Matrix in Network Theory," Microwave Res. Inst., Rep. R-366-54, PIB-30; 1954.

gated in the $-x$ direction is $(\frac{1}{2})bb^*$. These normalized waves have the dimensions of $\sqrt{P}$, and represent peak values.

In the usual transmission-line circuit, one of the component waves is excited by means of a generator, and so is known as the primary or incident wave; the other arises by reflection, or "scattering," from one or more discontinuities, or a mismatched termination. This separation of "cause and effect" is complete only in the case of a matched generator; otherwise, re-reflection takes place, creating an auxiliary primary wave.

The consideration of transmission lines in terms of traveling waves facilitates understanding of a phenomenon known as "standing waves." This phenomenon is nothing more than interference between the two component waves, producing successive stationary maxima and minima of the voltage and current along the line. The maxima occur where the two waves are in-phase, and the minima where they are out-of-phase. The ratio of the two is known as the voltage standing wave ratio, or vswr. If $a(x, t)$ is defined as the incident wave and $b(x, t)$ the reflected, the vswr, ρ, is given by:

$$\rho = \frac{|a| + |b|}{|a| - |b|}. \tag{7}$$

Thus if $b=0$, i.e., no reflection, then $\rho=1.0$. The importance of this quantity may be appreciated from the fact that microwave impedance measurements are usually made in terms of the magnitude and position of the standing waves produced.

Scattering Matrices

The use of traveling waves in describing transmission-line phenomena naturally leads to a scattering representation for the properties of transmission-line junctions. Whereas the impedance concept attempts to relate voltages and currents existing at the various junction ports, the scattering approach leads to a relationship between the incident and reflected waves at these ports. It seems logical to treat the incident waves as the independent quantities; we shall denote these as a_m, and express their contributions to the reflected or outward-traveling wave b_n at port n by a series of scattering coefficients, as:

$$\begin{aligned} b_1 &= S_{11}a_1 + S_{12}a_2 + \cdots S_{1n}a_n \\ b_2 &= S_{21}a_1 + S_{22}a_2 + \cdots S_{2n}a_n \\ &\vdots \\ b_n &= S_{n1}a_1 + S_{n2}a_2 + \cdots S_{nn}a_n. \end{aligned} \tag{8}$$

The justification for such a procedure is directly dependent upon the theory of linear superposition, just as is the corresponding impedance or admittance procedure.

Eqs. (8) may be formally reduced to a single equation by making use of a branch of mathematics known as matrix algebra. Observing the orderly nature of equations (8), we may group similar terms together in a form known as a "matrix," which is nothing more than an orderly array of such terms, and preserve the original equations intact by properly defining the rules for the manipulation of such matrices. Thus (8) may be written:

$$\begin{Bmatrix} b_1 \\ b_2 \\ \cdot \\ \cdot \\ b_n \end{Bmatrix} = \begin{Bmatrix} S_{11} & S_{12} \cdots S_{1n} \\ S_{21} & S_{22} \cdots S_{2n} \\ \cdot & \\ \cdot & \\ S_{n1} & S_{n2} \cdots S_{nn} \end{Bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ \cdot \\ \cdot \\ a_n \end{Bmatrix}, \tag{9}$$

which may be shortened to the symbolic matrix equation:

$$b = Sa. \tag{10}$$

The simplification is evident. The mechanics of handling matrix equations are found in numerous textbooks,[5] but need not concern us here; our primary interest is in the elements of the scattering matrix, the scattering coefficients themselves, and in certain theorems relating to them.

The simplest junction which we may consider is the one with the fewest ports, namely one; this may be identified with what is usually called a load or termination. In this case, the scattering matrix consists of a single term, S_{11}, whose definitions is obviously $S_{11}=b_1/a_1$. This, however, is just the usual definition of a reflection coefficient Γ, which is frequently used to characterize a microwave termination, and which is related to the load impedance Z_L (or admittance Y_L) by:

$$S_{11} = \Gamma_1 = \frac{b_1}{a_1} = \frac{Z_L - Z_c}{Z_L + Z_c} = \frac{Y_c - Y_L}{Y_c + Y_L}. \tag{11}$$

This is also the quantity which is plotted in polar coordinates on the familiar Smith chart. Thus a matched load, $Z_L=Z_c$, is represented by $S_{11}=\Gamma_1=0$, emphasizing the fact that a matched load by definition produces no reflection.

In order to understand the physical significance of the scattering-matrix elements which represent a multi-port junction, one need only consider a special case for which the scattering equations reduce to a simplified form. Thus if power is fed into a multi-port junction from the nth port, and all other ports are connected to matched loads, a_n will be the only incident wave, and the scattered or reflected waves emerging from each port will be, from (8):

$$\begin{aligned} b_1 &= S_{1n}a_n \\ b_2 &= S_{2n}a_n \\ &\vdots \\ b_n &= S_{nn}a_n. \end{aligned} \tag{8a}$$

Thus it is apparent that S_{nn} is the reflection coefficient seen looking into the nth port, with all others terminated in matched loads, while $S_{mn} (m \neq n)$ represents the

[5] See L. A. Pipes, "Applied Mathematics for Engineers & Physicists," McGraw-Hill Book Publishing Co., Inc., New York, N. Y.; 1946.

amplitude of the wave coupled out of the mth port for a unit incident wave at port n, under the same matched conditions. Since a normalized representation is being used, the corresponding coupled power relation may be written:

$$\frac{P_m\text{ (out)}}{P_n\text{ (in)}} = \frac{\frac{1}{2}b_m b_m^*}{\frac{1}{2}a_n a_n^*} = S_{mn}S_{mn}^* = |S_{mn}|^2. \quad (12)$$

Conditions Imposed upon the Scattering Matrix

Inasmuch as a scattering matrix is intended to represent the properties of a physical microwave junction, certain relationships exist within the scattering matrix as a result of the familiar laws of reciprocity and conservation of energy. Actually, the fact that the scattering matrix can be derived from the familiar impedance or admittance matrix for the same junction assures that this is so. This derivation can be developed[2] from the definition of the normalized component waves in (6) in terms of terminal voltages and currents, to yield the relationship:

$$S = (Z - 1)(Z + 1)^{-1} = (1 - Y)(1 + Y)^{-1}. \quad (13)$$

A slightly modified form would result from the use of non-normalized waves in (4).

The condition of reciprocity requires that the Z and Y matrices be symmetrical, and consequently S must be symmetrical from (13). This is represented by $S_{mn}=S_{nm}$. This is true, however, only for a normalized representation as in (6); the simplification which results from a symmetrical scattering matrix is therefore the justification for the normalization.

Conservation of energy as applied to a transmission line junction may be expressed in a more general form from Poynting's energy theorem for a periodic field[6] as:

$$\sum_n V_n I_n^* = 2P + 4j\omega(W_H - W_E), \quad (14)$$

which in traveling-wave form, from (4), becomes

$$\sum_n (a_n + b_n)(a_n^* - b_n^*) = 2P + 4j\omega(W_H - W_E), \quad (15)$$

where P is the power dissipated in the junction, and W_H and W_E are the average stored magnetic and electric energies, respectively. The real part of (15) is:

$$\sum_n (a_n a_n^* - b_n b_n^*) = 2P. \quad (16)$$

This may be expressed in matrix notation as:

$$\tilde{a}(1 - SS^*)a^* = 2P. \quad (17)$$

The requirement that $P \geqq 0$ imposes the condition:

$$\det(1 - SS^*) \geqq 0. \quad (18)$$

For the special case of a lossless junction, which is frequently approximated in practice, $P=0$, and

$$1 - SS^* = 0 \quad (19)$$

[6] Montgomery, Dicke and Purcell, *op. cit.*, pp. 132, 139, 148.

or

$$S^{-1} = S^* = \tilde{S}^*. \quad (20)$$

This is the definition of what is known as a unitary matrix, which has the special property that:

$$\sum_K S_{Km}S_{Kn}^* = \delta_{mn} = \begin{cases} 1 \text{ for } m = n \\ 0 \text{ for } m \neq n. \end{cases} \quad (21)$$

For $m=n$, this becomes:

$$\sum_K S_{Km}S_{Km}^* = \sum_K |S_{Km}|^2 = 1, \quad (22)$$

which can be identified from (12) as just the condition for conservation of energy.

Applications of Scattering Matrices

Section of Transmission Line

Suppose we wish to obtain the scattering matrix of a two-port junction consisting of a section of uniform lossless transmission line of length L, as shown in Fig. 1.

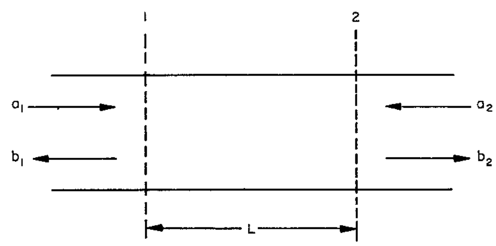

Fig. 1—Transmission-line section.

This is assumed to be a continuation of a similar transmission line connected to both pairs of terminals. The terminal planes are defined as shown. From the previously-indicated solution to the general transmission-line equations, it is apparent that a_1 and b_2 are related by a simple factor of the form $e^{-j\beta L}$, neglecting attenuation, and similarly a_2 and b_1. In fact

$$b_2 = e^{-j\beta L}a_1 \quad \text{and} \quad b_1 = e^{-j\beta L}a_2, \quad (23)$$

so that the scattering matrix equation is as follows:

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} 0 & e^{-j\beta L} \\ e^{-j\beta L} & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}. \quad (24)$$

From the simplicity of this result, it is apparent that a change in the specified location of the terminal planes of an arbitrary junction will affect only the phase of the scattering coefficients of the junction. In particular, if terminal-plane n is moved away from the junction a distance L, each of the scattering coefficients S_{mn} or S_{nm} involving n will be multiplied by the factor $e^{-j\beta L}$, while S_{nn} will involve two such factors and will be changed by a factor $e^{-2j\beta L}$.

Lossless Two-Port Junction

Certain general statements can be made about any lossless two-port junction, regardless of its form, merely as a result of the conditions specified previously. Such a junction may be a transition between two types of transmission lines, as shown in Fig. 2 (opposite).

From reciprocity and (22) we may write:

$$|S_{11}|^2 + |S_{12}|^2 = 1$$
$$|S_{21}|^2 + |S_{22}|^2 = 1$$
$$S_{12} = S_{21}. \tag{25}$$

Consequently, we may immediately conclude that:

$$|S_{11}| = |S_{22}|. \tag{26}$$

Thus it is apparent that the reflection coefficient is the same looking into either terminal with a matched load on the other, since under these conditions,

$$\rho_1 = \frac{1}{1} \frac{+}{-} \frac{|S_{11}|}{|S_{11}|} = \frac{1 + |S_{22}|}{1 - |S_{22}|} = \rho_2. \tag{27}$$

Furthermore, the fraction of the power reflected is:

$$\frac{P_{\text{reflected}}}{P_{\text{incident}}} = |S_{11}|^2, \tag{28}$$

while the insertion loss due to reflection is given by:

$$L = -10 \log_{10}(1 - |S_{11}|^2) = -20 \log_{10} |S_{12}|. \tag{29}$$

This latter form is valid also for a lossy junction, and includes the dissipation loss.

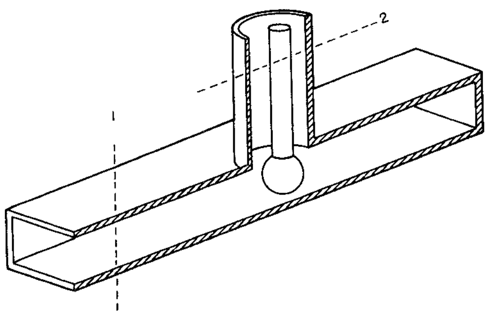

Fig. 2—Waveguide to coax adapter.

If now a load with reflection coefficient Γ_2 is connected to terminals 2,

$$\Gamma_2 = \frac{a_2}{b_2}, \quad \text{or} \quad a_2 = \Gamma_2 b_2, \tag{30}$$

and (8) can be solved simultaneously to yield the input reflection coefficient

$$\frac{b_1}{a_1} = S_{11} + \frac{S_{12}^2\Gamma_2}{1 - S_{22}\Gamma_2}. \tag{31}$$

These relations are also completely general, and hold for a lossy junction as well.

The use of a sliding mismatch for evaluating junction parameters is easily understood from the above relations. A sliding mismatch is merely a slightly mismatched termination whose reflection coefficient can be varied, in phase only, by sliding along the transmission line. From (31), it can be seen that as the phase of Γ_2 is changed, the input reflection coefficient will exhibit maxima and minima corresponding to the in-phase and out-of-phase conditions of the second term with respect to S_{11} (it must be assumed that $S_{22}\Gamma_2 \ll 1$). Under these conditions,

$$\left(\frac{b_1}{a_1}\right)_{\max} \cong S_{11} + |S_{12}^2\Gamma_2|$$
$$\left(\frac{b_1}{a_1}\right)_{\min} \cong S_{11} - |S_{12}^2\Gamma_2|, \tag{32}$$

from which S_{11}, $|S_{12}|$, and $|\Gamma_2|$ can be determined by (25).

Lossy Two-Port Junction

The scattering matrix for a lossy junction is not unitary, so the relations (21) and (22) do not hold, but may in general be replaced by:

$$\sum_K |S_{Km}|^2 < 1. \tag{33}$$

This is apparent from an extreme case suggested by Fig. 3, consisting of a resistance and a large condenser connected across a transmission line one-quarter wavelength apart. If $R = Z_c$ and $\omega C \gg Z_c$ the condenser will appear like a short-circuited quarter-wave stub across R, and will have very little effect; thus the junction will appear well-matched at terminals 1 regardless of the termination on 2, so $S_{11} \ll 1$ and $S_{12} \ll 1$. The input at 2 will be essentially a short-circuit, regardless of the termination on 1, so $S_{22} \cong -1$ and $S_{21} \ll 1$. Apparently, then,

$$|S_{11}| \neq |S_{22}|$$
$$|S_{11}|^2 + |S_{12}|^2 \ll 1$$
$$|S_{21}|^2 + |S_{22}|^2 \cong 1. \tag{34}$$

However, the reciprocity relation $S_{12} = S_{21}$ still holds.

This case is typical of a resistive microwave network, and is indicative of the fact that such a network may not be equally well-matched in both directions, such as a resistance card tapered on one end only.

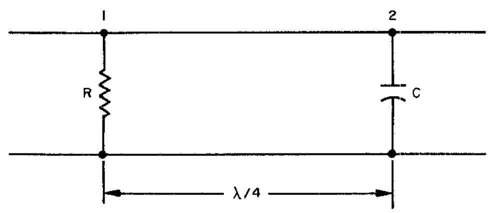

Fig. 3—Lossy two-port junction.

Multi-Port Lossless Junctions

An extension of the scattering representation to multiport junctions, together with full use of the reciprocity theorem, the unitary relations of (21) as applicable to a lossless junction, and conditions resulting from physical symmetry, will lead to a remarkable array of results without further information about the junction. The degree of losslessness and symmetry are frequently sufficient to justify the use of such assumptions, which greatly simplify the analytical results.

A very interesting application of these techniques can be made with regard to a waveguide "Magic-Tee," or side-outlet tee.[7] This device consists of a combination of

[7] C. G. Montgomery, "Technique of Microwave Measurements," Radiation Lab. Ser., vol. 11, McGraw-Hill Book Co., Inc., New York, N. Y.; 1947.

an E-plane and an H-plane waveguide tee arranged as shown in Fig. 4. The properties of this junction may be represented by a 4×4 scattering matrix, with a total of 16 terms. The number of independent terms is reduced to 10 by the reciprocity relations. Furthermore, symmetry of the junction requires that $S_{13}=S_{23}$, $S_{14}=-S_{24}$, and $S_{11}=S_{22}$, leaving the following seven terms:

$$\begin{Bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{12} & S_{11} & S_{13} & -S_{14} \\ S_{13} & S_{13} & S_{33} & S_{34} \\ S_{14} & -S_{14} & S_{34} & S_{44} \end{Bmatrix}. \quad (35)$$

From (21) with $m=3$, $n=4$, and using the above matrix (35),

$$S_{13}S_{14}^* - S_{13}S_{14}^* + S_{33}S_{34}^* + S_{34}S_{44}^* = 0. \quad (36)$$

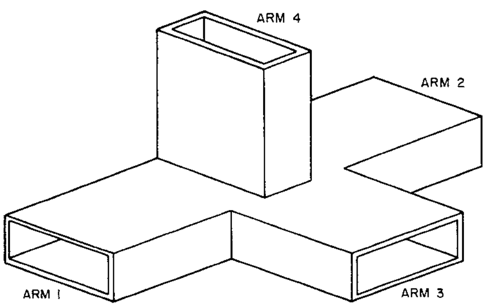

Fig. 4—Waveguide magic-tee.

Since S_{33} and S_{44} are inherently independent, this equation can be satisfied only with $S_{34}=0$, showing that there is no direct cross-coupling between the side arms of the tee. Because of this independence between the side arms, separate matching structures may be employed in each of these arms to produce (at least for a single frequency) matched inputs ($S_{33}=S_{44}=0$). Under these conditions, and with $S_{34}=0$, applying (22) to the matrix (35) yields:

$$\begin{aligned} |S_{11}|^2 + |S_{12}|^2 + |S_{13}|^2 + |S_{14}|^2 &= 1 \\ 2|S_{13}|^2 &= 1 \\ 2|S_{14}|^2 &= 1. \end{aligned} \quad (37)$$

From this we may conclude that $|S_{13}|=|S_{14}|=1/\sqrt{2}$, and $S_{11}=S_{22}=S_{12}=0$. Thus matching of the side arms automatically produces matching of the symmetrical arms, as well as decoupling between them. These are the conditions which are assumed to prevail in a truly "magic" tee, also known as a bi-conjugate network. The scattering matrix thus reduces to,

$$\frac{1}{\sqrt{2}}\begin{Bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{Bmatrix}, \quad (38)$$

assuming the proper choice of terminal planes.

Further use can be made of the scattering representation to evaluate, as in (31), the outputs from the various arms with arbitrary terminations on each, as necessary for understanding the performance of the unit under actual operating conditions, such as for use in an impendance bridge or a balanced mixer.

Measurement of the Scattering Coefficients of a Microwave Junction

Measurement of the actual values of the scattering matrix elements representing a particular microwave junction, especially a multi-port junction, may be quite difficult. The procedure usually used is basically the same as that employed at lower frequencies, consisting of measuring the input impedances or reflection coefficients at one or more ports with specified terminations on the other. Substitution of the results of such measurements into a set of equations similar to (31) yields the desired parameters. The process of finding the scattering coefficients is considerably simplified if well-matched loads are available for all ports; this however, may be an unknown condition in itself, especially in the case of a transition from a measuring section to another transmission line. A short-circuit termination may also be used, since a good short-circuit may be easily achieved in most types of transmission line; and if it is located a specific distance from the actual terminal plane of the junction, the short-circuit may be used to simulate any magnitude of reactive termination.

A procedure known as "Deschamps' Method"[8,9] has been devised for measuring the scattering coefficients of a two-port junction using four short-circuit terminations separated a quarter-wavelength apart in pairs (or alternatively, a sliding short-circuit set at these positions). Measurement of the input reflection coefficients at one port only with each of the four terminations on the other port are required, and the results are obtained from a very simple graphical construction. The method is easily applied to the direct measurement of unknown impedances through such a junction, and may be extended to multi-port junctions with additional measurements. It also permits a simple evaluation of experimental errors from the results of additional measurements.

Another method for the measurement of junction parameters which has been developed is known as the "Weissfloch Tangent" technique.[10] Its use results in the more direct evaluation of the impedance matrix or the equivalent circuit elements. The relative merits of the two methods have been discussed elsewhere.[11]

[8] G. A. Deschamps, "Determination of reflection coefficients and insertion loss of a waveguide junction," *Jour. Appl. Phys.*, vol. 24, pp. 1046–1050; August, 1953.

[9] J. E. Storer, L. S. Sheingold, and S. Stein, "A simple graphical analysis of a two-port waveguide junction," Proc. IRE, vol. 41, pp. 1004–1013; August, 1953.

[10] Marcuvitz, op. cit., pp. 130–138.

[11] L. B. Felsen and A. A. Oliner–S. Stein, L. S. Sheingold, and J. E. Storer, Proc. IRE, vol. 42, pp. 1447–1448; September, 1954.